ON THE MOTION OF A RIGID STRIP-MASS LYING ON AN ELASTIC HALF-SPACE AND EXCITED BY A SEISMIC WAVE

(O DVIZHENII POD DEISTVIEM SEISMICHESKOI VOLNY ZHESTKOI MASSIVNOI POLOSY, LEZHASHCHEI NA UPRUGOM POLUPROSTRANSTVE)

PMM Vol.26, No.6, 1962, pp. 1043-1058

L. M. FLITMAN
(Moscow)

(Received September 6, 1962)

We consider a rigid strip-mass of finite width and infinite length which lies on an elastic half-space. Friction between the strip and the half-space is assumed absent.

The motion of the strip is excited by a plane wave (longitudinal or transverse) which emerges from the depths of the half-space at an arbitrary angle, but in such a way that the normal to the front is perpendicular to the edge of the strip. The problem considered is plane. In this paper the parameters of small motion of the strip are determined, i.e. the vertical displacement of its middle and its angle of rotation.

To determine the resultant force and moment of the stresses under the strip, the mixed boundary value problem for an elastic half-plane is solved. In this solution the results of [1] are used.

A neighboring problem for the acoustical case was examined in [2,3].

For times less than $2l/a$ (a is the velocity of longitudinal waves in the medium, $2l$ is the width of the strip) the results that are obtained are simple.

1. Let an elastic body occupy the half-space $y \geqslant 0$. Its motion is described by the dynamic equations of Lamé

$$(\lambda + \mu)\,\mathrm{grad\,div}\,\mathbf{u} + \mu\Delta\mathbf{u} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}$$

Here λ and μ are the Lamé constants, ρ is the density, $\mathbf{u}$ is the displacement vector, and u and v are its x and y components.

The initial values are zero. On the boundary $y = 0$ we prescribe the conditions.

$$\tau_{xy} = 0 \quad (-\infty < x < \infty), \qquad \sigma_{yy} = 0 \quad (x < 0), \qquad v = f(x, t) \quad (x > 0) \tag{1.1}$$

Here $f(x, t)$ is a given function. This is the boundary value problem for a semi-infinite stamp [1].

The following relation [1] exists between the formula obtained by using the Laplace transformation for σ_{yy} in x and t at $y = 0$ and the formula obtained by using the same transformation for v at $y = 0$ and $x > 0$

$$\sigma(q, p) = -2(1 - \gamma^2)\rho bpK(-s)L(q, p) \qquad \left(b^2 = \frac{\mu}{\rho},\ a^2 = \frac{\lambda + 2\mu}{\rho},\ \gamma = \frac{b}{a}\right) \tag{1.2}$$

Here q and p are the Laplace transform parameters with respect to x and t

$$K(s) = \frac{\vartheta - s}{\sqrt{\gamma - s}} e^{-g(s)}, \qquad g(s) = \frac{1}{\pi}\int_{\gamma}^{1} \varphi(\xi) \frac{d\xi}{\xi - s} \qquad \left(s = \frac{bq}{p}\right) \tag{1.3}$$

$$\varphi(\xi) = \tan^{-1}\left\{\frac{4\xi^2\sqrt{(1 - \xi^2)(\xi^2 - \gamma^2)}}{(2\xi^2 - 1)^2}\right\}$$

where ϑ is the root of the Rayleigh equation

$$G(s) \equiv (2s^2 - 1)^2 + 4s^2\sqrt{(1 - s^2)(\gamma^2 - s^2)} = 0 \tag{1.4}$$

Thus the function $K(s)$ is regular for $\operatorname{Re} s < \gamma$

$$L(q, p) = -\frac{1}{2\pi i}\int_{l} K(s_1) V(\zeta, p) \frac{d\zeta}{\zeta - q} \qquad \left(s_1 = \frac{b\zeta}{p}\right) \tag{1.5}$$

where l is a straight line in the complex plane q, parallel to the imaginary axis and located between the imaginary axis and the point $q = p/a$

$$\sigma(q, p) = \iint_0^\infty \sigma_{yy}(x, t)\, e^{-qx-pt}\, dxdt, \qquad V(q, p) = \iint_0^\infty v(x, t)\, e^{-qx-pt}\, dxdt \tag{1.6}$$

Hence, with the aid of formulas (1.2) to (1.6) one may obtain an expression for $\sigma(q, p)$, corresponding to the boundary value problem (1.1).

We examine three particular cases of this problem

$$v = f(t), \qquad v = x\alpha(t), \qquad v = f(t - \theta x) \tag{1.7}$$

The first case corresponds to the progressive penetration of a stamp, the second to the rotation through a small angle α with respect to the edge, and the third, as will be explained below, to the reflection of a plane wave from the boundary under mixed boundary conditions.

Let us examine the first case, In agreement with (1.6), we obtain

$$V(q, p) = \frac{F(p)}{q} \qquad \left((F(p) = \int_0^\infty f(t) e^{-pt} dt \right)$$

Making use of the fact that $K(s_1)$ is regular for Re $\zeta < a^{-1}$ Re p, we have from (1.5), by virtue of the residue theorem

$$L(q, p) = k_0 \frac{F(p)}{q} \qquad \left(k_0 = K(0) = \frac{1}{\sqrt{2\gamma(1-\gamma^2)}} \right)$$

Now, from (1.2) one may obtain

$$\sigma(q, p) = -\frac{a\rho}{k_0} pF(p) \frac{K(-s)}{q}$$

We apply to $\sigma(q, p)$ the Laplace transform inversion formula. Then

$$\sigma_{yy}{}^\circ(x, p) = -\frac{a\rho}{k_0} pF(p) \frac{1}{2\pi i} \int_{l_0} K(-s) e^{qx} \frac{dq}{q}$$

The contour l_0 is parallel to the naginary axis of the complex plane q and lies in the right half-plane. y means of the residue theorem, the expression that has been obtained r .y be brought to the form

$$\sigma_{yy}{}^\circ(x, p) = -a\rho pF(p) - \frac{a\rho}{k_0} pF(p) \frac{1}{2\pi i} \int_{l_0} K(-s) e^{qx} \frac{dq}{q} \tag{1.8}$$

If the contour l_1 is "placed" onto the cut [1] introduced into the s plane to separate out a single-valued branch of $K(-s)$, we obtain

$$\sigma_{yy}{}^\circ(x, p) = -a\rho pF(p) + \frac{a\rho}{k_0} pF(p) \int_1^\infty N(s) \exp\left\{-\frac{spx}{b}\right\} ds \tag{1.9}$$

Here

$$N(s) = \frac{1}{\pi} \frac{\vartheta - s}{\sqrt{s-\gamma}} e^{-g(s)} \cos\varphi(s) \qquad (\gamma \leqslant s \leqslant 1),$$
$$N(s) = \frac{1}{\pi} \frac{\vartheta - s}{\sqrt{\gamma - s}} e^{-g(s)} \qquad (s \geqslant 1) \tag{1.10}$$

The functions $g(s)$ and $\varphi(s)$ are given by formulas (1.3), and the integral defining $g(s)$ for $\gamma \leqslant s \leqslant 1$ is to be understood in the principal value sense.

Carrying out the inversion with respect to p, we obtain finally

$$\sigma_{yy}(x, t) = -a\rho f'(t) \qquad (at < x) \tag{1.11}$$

$$\sigma_{yy}(x, t) = -a\rho f'(t) + \frac{a\rho}{k_0}\frac{\partial}{\partial t}\int_{\gamma}^{bt/x} N(s) f\left(t - \frac{sx}{b}\right) ds \qquad (at > x)$$

The first term in this expression gives the stress in the plane wave which would be generated underneath the stamp if the stamp extended over the entire boundary, whereas the second term gives the stress in waves which propagate from the edge of the stamp. The fronts of these waves are shown in Fig. 1. Let us compute the magnitude of the resultant and moment of these stresses.

It can be shown that the resultant is proportional to the displacement of the stamp, $f(t)$, since for $f(t) = t$ the second term in (1.11) depends only on bt/x. This was kindly communicated to the author by V.P. Krysin.

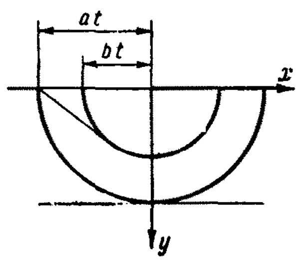

Fig. 1.

We denote the resultant of these stresses by $R_1(t)$ and its transform by $R_1^{\circ}(p)$. Then from (1.8) we obtain

$$R_1^{\circ}(p) = -\frac{a\rho}{k_0} pF(p)\int_0^{\infty} dx \frac{1}{2\pi i}\int_{l_1} K(-s)\, e^{qx}\frac{dq}{q}$$

Integrating with respect to x under the integral sign, we note that the remaining integral reduces to the evaluation of the residue at the origin. As a result, we obtain

$$R_1^{\circ}(p) = -\tfrac{1}{2} C_1(\gamma)(\lambda + 2\mu) F(p) \qquad \left(C_1(\gamma) = 2\gamma\frac{k_1}{k_0}\right) \tag{1.12}$$

Here and in the sequel $k_0, k_1, k_2, \ldots$ denote the coefficients in the expansion of $K(s)$ in the neighborhood of the origin

$$K(s) = k_0 + k_1 s + k_2 s^2 + k_3 s^3 + \ldots \tag{1.13}$$

From (1.12) it follows

$$R_1(t) = -\tfrac{1}{2} C_1(\gamma)(\lambda + 2\mu) f(t) \tag{1.14}$$

Analogously, we compute the moment $M_1(t)$ of these stresses with respect to the edge of the stamp

$$M_1(t) = \frac{1}{2} D_1(\gamma)(\lambda + 2\mu) a \int_0^t f(\tau)\, d\tau \qquad \left(D_1(\gamma) = 2\gamma^2 \frac{k_2}{k_0}\right) \tag{1.15}$$

We examine next the second of cases (1.7). In agreement with (1.5) and (1.6) we have

$$L(q, p) = A(p)\left(\frac{k_0}{q^2} + \frac{bk_1}{pq}\right) \qquad \left(A(p) = \int_0^\infty \alpha(t) e^{-pt}\, dt\right)$$

Proceeding in the same manner as before, we obtain, analogous to (1.8)

$$\sigma_{yy}^\circ(x, p) = -a\rho pA(p)x - \frac{a\rho}{k_0^2} pA(p) \frac{1}{2\pi i} \int_{l_1} \left(\frac{k_0}{q^2} + \frac{bk_1}{pq}\right) K(-s) e^{qx}\, dx \tag{1.16}$$

or, deforming the contour l_1, we find

$$\sigma_{yy}^\circ(x,p) = -a\rho pA(p)x + \frac{\lambda + 2\mu}{k_0^2} A(p) \int_\gamma^\infty \left(k_1 - \frac{k_0}{s}\right) N(s) \exp\left\{-\frac{spx}{b}\right\} ds \tag{1.17}$$

Inverting with respect to p, we find the distribution of stress σ_{yy} under the base of the stamp

$$\sigma_{yy}(x, t) = -a\rho x\alpha'(t) \qquad (at < x) \tag{1.18}$$

$$\sigma_{yy}(x, t) = -a\rho x\alpha'(t) + \frac{\lambda + 2\mu}{k_0^2} \int_\gamma^{bt/x} \left(k_1 - \frac{k_0}{t}\right) N(s)\, \alpha\left(t - \frac{sx}{b}\right) ds \qquad (at > x)$$

Here, as in formula (1.11), the first term describes the stress which would be produced under the stamp if the stamp occupied the entire boundary, and the second term gives the stress in waves that propagate from the edge of the stamp. In this case as well, the resultant and the moment with respect to $x = 0$ of these stresses may be calculated. For this it is convenient to use formula (1.16)

$$R_2(t) = \frac{1}{2} C_2(\gamma)(\lambda + 2\mu) a \int_0^t \alpha(\tau)\, d\tau \qquad \left(C_2 = 2\gamma^2\left[\left(\frac{k_1}{k_0}\right)^2 - \frac{k_2}{k_0}\right]\right)$$

$$M_2(t) = -\frac{1}{2} D_2(\gamma)(\lambda + 2\mu) a^2 \int_0^t (t - \tau)\, \alpha(\tau)\, d\tau \qquad \left(D_2 = 2\gamma^3\left(\frac{k_3}{k_0} - \frac{k_1 k_2}{k_0^2}\right)\right) \tag{1.19}$$

In the third case of (1.7) we shall assume

$$f(\xi) = 0 \quad \text{for } \xi > 0, \qquad \theta^{-1} > a$$

Then some calculations give

$$\sigma_{yy}{}^{\circ}(x,p) = -\frac{G(b\theta)}{\sqrt{1-a^2\theta^2}}\, a\rho p F(p)\, e^{-p\theta x} + $$

$$+\frac{K(-b\theta)}{k_0{}^2}\, a\rho p F(p) \int_{\gamma}^{\infty} N(s) \exp\left\{-\frac{spx}{b}\right\} \frac{s\,ds}{s-b\theta} \qquad (1.20)$$

The stress distribution is obtained by inverting this formula with respect to p.

$$\sigma_{yy}(x,t) = 0 \qquad (t < \theta x) \qquad (1.21)$$

$$\sigma_{yy}(x,t) = -\frac{G(b\theta)}{\sqrt{1-a^2\theta^2}}\, a\rho f'(t-\theta x) \qquad (at < x < t/\theta)$$

$$\sigma_{yy}(x,t) = -\frac{G(b\theta)}{\sqrt{1-a^2\theta^2}}\, a\rho f'(t-\theta x) + \frac{K(-b\theta)}{k_0{}^2}\, a\rho \frac{\partial}{\partial t} \int_{\gamma}^{bt/x} N(s) f\left(t-\frac{sx}{b}\right) \frac{s\,ds}{s-b\theta}$$

$$(at > x)$$

Proceeding as above, it is possible to compute the resultant, $R_3(a\theta, t)$, and the moment $M_3(a\theta, t)$ (with respect to $x = 0$), of the stresses (1.21)

$$R_3(a\theta,t) = -C_3(a\theta,\gamma)(\lambda+2\mu) f(t) \qquad \left(C_3(a\theta,\gamma) = \frac{\gamma^2 G(b\theta)}{b\theta\sqrt{\gamma^2-b^2\theta^2}} \frac{k_0}{K(b\theta)}\right) \quad (1.22)$$

$$M_3(a\theta,t) = D_3(a\theta,\gamma)(\lambda+2\mu)\, a \int_0^t f(\tau)\,d\tau \qquad \left(D_3(a\theta,\gamma) = \frac{\gamma}{b\theta} C_3(a\theta,\gamma)\left(1+\frac{k_1}{k_0} b\theta\right)\right)$$

We may formulate a problem whose boundary conditions have the same form as the third case of (1.7).

For $x \geqslant 0$, let a rigid strip lie on an elastic half-plane, with points in the strip medium being prevented from moving vertically. In addition, assume the absence of friction between the strip and the elastic medium. Thus, for $y = 0$ and $x \geqslant 0$ points of the boundary that are contiguous with the strip, the conditions may be formulated as

$$\tau_{xy} = 0, \qquad v = 0$$

The remaining portion of the boundary $y = 0$, $x < 0$ is free of stresses.

Let a plane wave be incident on the boundary, approaching the plate from the direction of the free portion of the boundary and impinging on the edge at $t = 0$.

For $t < 0$, the motion of the medium may be determined by the known laws for the reflection of plane waves.

For $t > 0$, the combination of incident and reflected plane waves from the free boundary satisfies the boundary conditions for $x < 0$, and likewise the condition $\tau_{xy} = 0$ for $x > 0$. For $x > 0$, the condition that the vertical displacement should vanish is not fulfilled, and the displacement in this combination of waves may be expressed in the form $v = f(t - \theta x)$, where θ^{-1} is the velocity of the trace of the wave fronts along the boundary. Therefore for $t > 0$ it is necessary to supplement the solution that we already have by the solution of a mixed boundary value problem for an elastic half-plane, which, for $y = 0$ satisfies the conditions

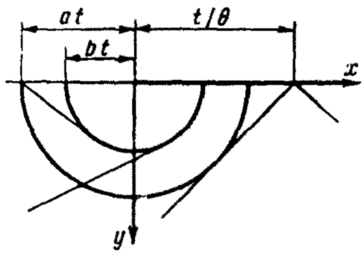

Fig. 2.

$$\tau_{xy} = 0 \quad (-\infty < x < \infty), \qquad \sigma_{yy} = 0 \quad (x < 0)$$
$$v = -f(t - \theta x) \quad (x > 0) \qquad (1.23)$$

But these are just the conditions for which the stresses (1.21) were computed. In such a case, the first term of this formula, taken with opposite sign, may be interpreted as the stress caused by the incidence on the boundary of an arbitrary plane wave, when the boundary conditions $\tau_{xy} = 0$ and $v = 0$ are satisfied. One may not overlook the fact here that $f(t - \theta x)$ is the total vertical displacement on the free boundary when the same plane wave is incident on it.

By the use of the theory of reflection of plane waves, it may be verified, independently of the above argument, that the incidence of such an arbitrary plane wave produces a total displacement of the free boundary and a normal stress of the "semirigid" boundary that are related in just this way.

As in formulas (1.11) and (1.16), the second term gives a stress in the waves propagating from the edge of the plate and describes a phenomenon close to the phenomenon of diffraction. The fronts of all the waves that are produced are shown in Fig. 2.

Another modification of this same problem may be given which also leads to the solution (1.21). On the boundary of the elastic half-plane, let a load move in the positive x-direction

$$\sigma_{yy} = -p(t - \theta x) \quad (\theta^{-1} > a), \qquad \tau_{xy} = 0, \qquad p(\xi) = 0 \quad \text{for } (\xi < 0) \qquad (1.24)$$

At $t = 0$ the load reaches the edge of the rigid strip and continues to move farther along it. The problem consists of determining the motion of the medium and, which is more important, the stresses produced under the strip. The solution, as in the first modification of the problem, is sought as the sum of two solutions.

The first is represented in the form of two plane waves, longitudinal and transverse, proceeding from the boundary downward into the medium, with the vertical displacements on the boundary being given by

$$v(\xi) = \frac{\sqrt{\gamma^2 - b^2\theta^2}}{b\theta G(b\theta)} \int_0^{\xi} p(\tau)\, d\tau \qquad (\xi = t - \theta x) \tag{1.25}$$

The conditions (1.23) obtain in the case of the second solution, where v is given by (1.25). It is already clear that the formula (1.21) gives an expression for the stresses on the boundary in the case of the second solution. The actual stress under the plate is given by the sum of the stresses of the two solutions, i.e. formula (1.21), with the first term omitted, and $f(\xi)$ taken from (1.25) with opposite sign.

We list here the values of the coefficients C_1, C_2, D_2 from (1.12) and (1.19), as functions of γ

TABLE 1.

$\gamma =$	0.1	0.2	0.3	0.4	0.5	0.6
$C_1 =$	−0.6839	−0.3751	−0.09906	0.1340	0.1340	0.4343
$C_2 =$	−0.3479	−0.3379	−0.2467	−0.1131	0.02397	0.1213
$D_2 =$	0.1824	0.3465	0.3665	0.2889	0.1702	0.06726

We note that as γ decreases the coefficients C_0 and C_2 change sign. This indicates that tensile stresses dominate in the wave produced on the edge of the stamp. Hence, for penetration at these values of γ, it is possible for the medium to "peel-off" from the bottom of the stamp. At $\gamma = 0$, a calculation yields $C_1 = -1$, which agrees with the solution of the corresponding acoustic problem [2, 4].

2. We consider the following problem. A rigid strip-mass of infinite length and width $2l$ lies on an elastic half-space $y \geqslant 0$. Friction between the strip and the elastic medium is absent, and vertical displacements of points of the medium coincide with the displacements of corresponding points of the strip.

Assume now that at $t = 0$ a normal plane longitudinal wave from below is incident on the surface of the half-space. It is required to find the motion of the strip under the action of the wave. The solution will be constructed for $t < 2l/a$.

We place the origin of coordinates at the center of the plate, with the y-axis directed downwards and the x-axis directed perpendicular to the edge of the plate. The displacements in the indicent waves are

are denoted by $v = v_0(t + y/a)$.

The mass per unit length of the plate is denoted by m, the resultant force of the normal stresses under the plate by R, and the displacements of the plate itself by $v(t)$. Then

$$mv''(t) = R \tag{2.1}$$

The motion of the elastic medium excited by the incidence of the wave on the boundary and the oscillations of the plate may be represented as the sum of two motions: the first is the motion which would be excited in the absence of the plate (reflection of a plane wave from a free boundary), and the second is the motion that would occur in the half-space if it were penetrated by the strip according to the as yet unknown law $v_1(t)$.

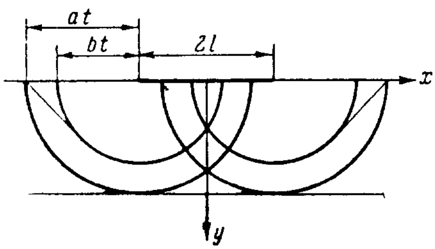

Fig. 3.

Then

$$v(t) = 2v_0(t) + v_1(t) \tag{2.2}$$

The fronts of all the waves that are produced are sketched in Fig. 3.

Let us compute R. The first motion does not contribute to R. Therefore to compute R it is necessary to use the solution of the dynamic stamp problem. However, if the time is limited to $t < 2l/a$, it is sufficient to know the solution for the semi-infinite stamp penetrating the half-plane according to $v_1(t)$. Then, using (1.11) and (1.12), we obtain

$$R = -2a\rho l v'_1(t) - C_1(\lambda + 2\mu) v_1(t)$$

Substituting this and (2.2) into (2.1), we will have

$$mv''_1 + 2a\rho l v''_1 + C_1(\lambda + 2\mu) v_1 = -2mv''_0$$

or

$$v''_1 + 2Mnv'_1 + C_1 M n^2 v_1 = -2v''_0 \qquad (M = \rho l^2/m,\ n = a/l) \tag{2.3}$$

The initial conditions are formulated in the following way

$$v'_1(0) + 2v'_0(0) = 0, \qquad v_1(0) + 2v_0(0) = 0$$

Seismic waves from earthquakes are usually of long period, whereas the interval of time for which the motion of the strip is described by equations (2.3) is not large. Hence it is convenient to consider two cases

$$v_0(t) = -\frac{p_0}{a\rho}t, \qquad v_0(t) = -\frac{p_0}{2a\rho T}t^2 \tag{2.4}$$

For the first of these the complete motion of the strip is given by the formula

$$v(t) = -\frac{2p_0}{a\rho}\left(t - \frac{l}{\alpha a}e^{-M\tau}\sinh\alpha\tau\right) \qquad \left(\alpha = \sqrt{M^2 - MC_1},\ \tau = \frac{at}{l}\right) \tag{2.5}$$

Thus it is seen that the motion of the strip "relative to the wave" may be either aperiodic or, if $M < C_1$, a damped sinusoid.

The parameter M characterizes the ratio of a certain mass of the medium to the mass of the strip. If the strip is "light", the motion is aperiodic and the damping is large; the "heavier" the strip, the weaker the damping, whereas the frequency of the oscillation first increases and then diminishes.

For $M \geqslant 1$ (light strip) the maximum deviation from the motion of the free boundary will be attained within the time interval for which equation (2.3) holds. Namely, at time

$$t_* = \frac{l}{a}\frac{1}{2\alpha}\ln\frac{M+\alpha}{M-\alpha}$$

The lighter the strip, and the smaller t_*, the faster it will move, as the boundary would move in the absence of the strip.

As for the acceleration of the entire motion of the strip, its maximum value in the present case is attained at the instant of approach of the wave and is equal to

$$v_*'' = -4pl/m$$

Subsequently it approaches zero.

In the second case (2.4), the acceleration of the complete motion of the strip is

$$v''(t) = -\frac{p_0}{a\rho T}\left\{1 + \frac{1}{2\alpha}\left[(M-\alpha)e^{-(M-\alpha)\tau} - (M+\alpha)e^{-(M+\alpha)\tau}\right]\right\}$$

at the initial instant of time the acceleration of the strip is zero, and afterwards it begins to increase. If the strip is sufficiently light, its maximum is attained at the time

$$t_{**} = \frac{l}{\alpha a}\ln\frac{M+\alpha}{M-\alpha}$$

By the use of the foregoing formula, it is possible to compute its magnitude.

One last comment. If

$$v_0(t) = A\sin\omega t$$

then for sufficiently light strips, frequencies occur such that the strip breaks away from the half-space, i.e. the resultant of the stresses under the strip goes to zero and the velocity is negative.

3. We consider, as in Section 2, a rigid, homogeneous strip-mass of width $2l$ on an elastic half-space. We locate the coordinate system in the same way as in Section 2. We shall solve the problem of the motion of the strip when this motion causes a plane wave (completely longitudinal or transverse), incident at an arbitrary angle less than the critical angle (for a free boundary) and initially reaching the left edge of the strip at $t = 0$. The vertical displacements produced by the reflection of the plane wave from the free boundary $y = 0$ have the form

$$v = v_0(t - \theta x) \qquad (\theta^{-1} > a) \tag{3.1}$$

Under the action of the wave, the strip begins to move. However, since it is rigid and, by assumption, there is no separation, the vertical displacement of points of the medium bounded by the strip will be given by

$$v(x, t) = w(t) + x\alpha(t) \tag{3.2}$$

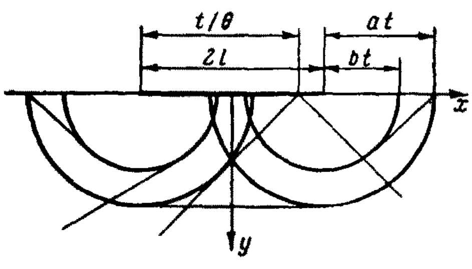

Fig. 4.

Here $w(t)$ is the displacement of the center of the strip and $\alpha(t)$ is the small angle of rotation of the strip.

The equations of motion of the strip are

$$mw''(t) = R(t), \qquad I\alpha''(t) = M(t) \tag{3.3}$$

Here m is the mass per unit length of the strip, I is the moment of inertia per unit length relative to the middle, $R(t)$ and $M(t)$ are the resultant and resultant moment (relative to the middle) of the stresses under the strip. The initial conditions for equations (3.3) are all zero.

In order to express $R(t)$ and $M(t)$ in terms of v_0, w and $\alpha(t)$ we represent the motion of the elastic medium as the sum of two motions: one excited by the penetration of the stamp into the elastic half-plane in accordance with (3.2), and the second excited by the reflection of a plane wave from the boundary under the following conditions

$$\tau_{xy} = 0 \quad (-\infty < x < \infty), \qquad \sigma_{yy} = 0 \quad (|x| > l), \qquad v = 0 \quad (|x| < l)$$

The wave fronts that are produced are sketched in Fig. 4.

If the solution of these problems is sought for $t < 2l\theta$, i.e. for times such that the trace of the incident wave has not passed over the entire width of the strip, then the results presented in Section 1 may be used.

Let us compute the resultant. The stresses excited by a small rotation of the strip do not contribute to the resultant. Using formulas (1.12) and (1.22), wherein it is necessary to set $f(t) = w(t)$ and $f(t) = -v_0(t)$, respectively, we obtain

$$R(t) = -2a\rho l w' - C_1(\lambda + 2\mu) w + C_3(\lambda + 2\mu) v_0 \tag{3.4}$$

In the computation of the moment of the stresses relative to the middle of the strip, it is necessary to take into consideration the fact that the stresses excited by a translational motion of strip give a zero moment.

As regards the stress moment excited by a small rotation, one should keep in mind that in the neighborhood of the left end the displacement may be represented as the sum of displacements excited by a translational displacement of the strip of magnitude $-\alpha(t)l$ and a rotation through the angle $\alpha(t)$; likewise, in the neighborhood of the right end as a translational displacement of $\alpha(t)l$ and a rotation through the angle $-\alpha(t)$.

We express the stress moment $M(t)$ in terms of $R_1(t)$, $R_2(t)$, $M_2(t)$, $R_3(a\theta, t)$, $M_3(a\theta, t)$ from (1.13), (1.19) and (1.22). In this it is necessary to set in (1.12) and (1.22)

$$f(t) = l\alpha(t), \qquad f(t) = -v_0(t)$$

respectively.

Then we obtain

$$M(t) = -\frac{4n\rho l\alpha'(t)}{3} + 2lR_1(t) - 2lR_2(t) - 2M_2(t) - lR_3(a\theta, t) + M_3(a\theta, t \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3) and setting $I = ml^2/3$, we obtain, after some transformations

$$w'' + 2Mnw' + C_1Mn^2w = C_3Mn^2v_0 \qquad (M = \rho l^2/m,\ n = a/l) \tag{3.6}$$

$$\alpha^{IV} + 4Mn\alpha''' + 3C_1Mn^2\alpha'' + 3C_2Mn^3\alpha' + 3D_2Mn^4\alpha = \\ = -3C_3Mn^2l^{-1}v_0'' - 3D_3Mn^3l^{-1}v_0' \quad (3.7)$$

The solutions of these equations must satisfy the initial conditions

$$w(0) = w'(0) = 0$$
$$\alpha(0) = \alpha'(0) = \alpha''(0) = \alpha'''(0) = 0, \quad \text{if} \quad v_0'(0) = 0 \quad (3.8)$$

This follows from the second of equations (3.3) and its initial conditions. The roots of the characteristic equation corresponding to (3.6) are

$$\lambda_{1,2} = -\frac{a}{l}(M \pm \sqrt{M^2 - MC_1}) \quad (3.9)$$

The roots of the characteristic equation corresponding to (3.7) may be represented in the form

$$\lambda_k = -\frac{a}{l}\nu_k$$

We list some values of ν_k as a function of M, for $\gamma = 1/\sqrt{3}$

TABLE 2.

M	ν_1	ν_2	$\nu_{3,4} = \varepsilon \pm i\omega$
0.5	1.659	0.05743	$0.1719 \pm i\,0.1493$
0.6	2.064	0.05739	$0.1391 \pm i\,0.1469$
0.7	2.468	0.05736	$0.1372 \pm i\,0.1454$
0.8	2.871	0.05734	$0.1359 \pm i\,0.1743$
1.0	3.675	0.05731	$0.1350 \pm i\,0.1429$
1.2	4.477	0.05729	$0.1329 \pm i\,0.1420$
1.5	5.679	0.05727	$0.1318 \pm i\,0.1411$

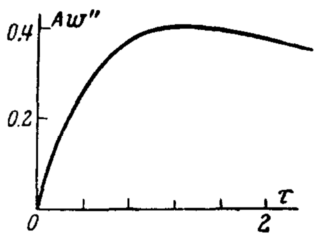

Fig. 5.

As an example we consider, as in Section 2, an incident longitudinal wave, characterized by a constant compressive stress $-p_0$ and a velocity of motion of the trace of the front along the boundary of $\theta^{-1} > a$.

Then

$$v_0(t) = -\frac{p_0}{a\rho}\frac{\sqrt{\gamma^2 - b^2\theta^2}}{\gamma G(b\theta)}t \quad (3.10)$$

The acceleration of translational motion will be

$$w''(t) = -\frac{\rho_0}{\rho l}\frac{C_4}{2\sqrt{M^2 - MC_1}}(e^{\lambda_1 t} - e^{\lambda_2 t}) \qquad \left(C_4 = \frac{\gamma k_0}{b\theta K(b\theta)}\right) \quad (3.11)$$

where λ_1 and λ_2 are given by (3.9).

For $M = 1$ and $\gamma = 1/\sqrt{3}$, the graph of the quantity $w''(t)$ as a function of $\tau = at/l$ is shown in Fig. 5. In addition, in Fig. 5 we have introduced the notation

$$A = -\frac{l\rho}{C_4 p_0}$$

Hence, equations (3.6) and (3.7) may be used only for $t \leqslant 2l\theta$, i.e. for times for which the trace of the incident wave has not reached the right edge of the strip. Thus in cases for which $2l\theta$ is small (close to normal incidence), one can only say that the acceleration increases monotonically.

We turn next to equation (3.7). In the example considered it is not possible to use conditions (3.8), since $v'_0(0) \neq 0$.

Taking this into consideration in the differentiation of (3.6) with respect to t, we obtain

$$\alpha(0) = \alpha'(0) = \alpha''(0) = 0, \qquad \alpha'''(0) = -3C_4 M n^2 l^{-1} v'_0(\theta)$$

The solution of (3.7) for such initial conditions and the indicated choice of $v_0(t)$ has the form

$$\alpha(t) = 3M\frac{p_0}{a^2\rho}\left\{C_4\alpha_0(\tau) + D_4\int_0^\tau \alpha_0(\xi)\,d\xi\right\} \qquad (3.12)$$

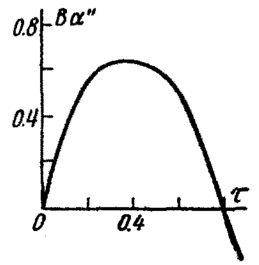

Fig. 6.

where

$$\tau = \frac{at}{l}, \qquad D_4 = \left(\frac{\gamma}{b\theta}\right)^2 \frac{k_0 + k_1 b\theta}{K(b\theta)}$$

$$\alpha_0(\tau) = A_1 e^{-\nu_1\tau} + A_2 e^{-\nu_2\tau} - 2e^{-\varepsilon\tau}(A_3\cos\omega\tau + A_4\sin\omega\tau)$$

The integration constants A_1, A_2, A_3 and A_4 are determined from the initial conditions.

The dependence of $\alpha''(\tau)$ for $M = 1$, $\gamma = 1/\sqrt{3}$, $a\theta = 0.5$ is shown in Fig. 6, where $B = \rho l^2/3D_4 p_0$.

4. In order to determine the motion of the strip for $t > 2l/a$, it is necessary to be able to form σ_{yy} for $y = 0$ and $x < 0$, when the following conditions are specified on the boundary of the elastic half-plane $y \geqslant 0$

$$\tau_{xy} = 0\ (-\infty < x < \infty), \quad v = 0 \quad (x < 0), \quad \sigma_{yy} = h(x, t) \quad (x > 0) \qquad (4.1)$$

Here $h(x, t)$ is a given function.

The solution of this problem may be obtained in the same manner as in [1] by means of the Wiener-Hopf-Fock method [5,6]. Therefore, we write

$$\sigma_2(q, p) = -K(s)\,L(q, p) \qquad \left(s = \frac{bq}{p}\right) \tag{4.2}$$

Here, as before, p and q are Laplace transform parameters with respect to x and t; the function $K(s)$ is given by (1.3), and L denotes a function that is regular in the complex q plane for $\operatorname{Re} q < a^{-1} \operatorname{Re} p$

$$L(q, p) = -\frac{1}{2\pi i}\int_l \frac{\sigma_1(\zeta, p)\,d\zeta}{K(s_1)(\zeta - q)} \qquad \left(s_1 = \frac{b\zeta}{p}\right) \tag{4.3}$$

The contour l is an infinite straight line, parallel to the imaginary axis, and lying in the right half-plane left of the point $\zeta = p/a$

$$\sigma_2(q, p) = \int_0^\infty e^{-pt}\,dt \int_{-\infty}^0 e^{-qx}\sigma_{yy}(x, t)\,dx, \qquad \sigma_1(q, p) = \iint_0^\infty e^{-pt-qx} h(x, t)\,dx\,dt \tag{4.4}$$

From the zero initial conditions it follows that σ_2 is regular for $\operatorname{Re} q < a^{-1} \operatorname{Re} p$, whereas σ_1 is regular for $\operatorname{Re} q > 0$.

In order to obtain $\sigma_{yy}(x, t)$, it is necessary to apply the inverse Laplace transform formulas relative to q and p to σ_2 from (4.4). The inversion with respect to q gives

$$\sigma_{yy}^{\circ}(x, p) = -\frac{1}{2\pi i}\int_{l_1} K(s)\,L(q, p)e^{qx}\,dq \tag{4.5}$$

The contour l_1 is a straight line lying between the imaginary axis of the q plane and the contour l. Integrating (4.5) with respect to x under the integral sign, we have

$$R^{\circ}(p) = \int_{-\infty}^0 \sigma_{yy}^{\circ}(x, p)\,dx = -\frac{1}{2\pi i}\int_{l_1} K(s)\,L(q, p)\,\frac{dq}{q}$$

However $K(s)$ and $L(q, p)$ are regular in the left-half-plane and their product tends to zero at infinity. Therefore the integral that has been obtained is equal to the residue at $q = 0$, and the transform of the resultant is

$$R^{\circ}(p) = -\frac{k_0}{2\pi i}\int_l \frac{\sigma_1(\zeta, p)\,d\zeta}{\zeta K(s_1)} \tag{4.6}$$

Analogously, for the transform of the moment $M(t)$, relative to $x = 0$, we have

$$M^{\circ}(p)=\frac{1}{2\pi i}\int_{l}\left(\frac{k_0}{\zeta^2}+\frac{k_1 b}{p\zeta}\right)\frac{\sigma_1(\zeta,p)}{K(s_1)}\,d\zeta \tag{4.7}$$

We consider the case when

$$h(x,t)=h(t-\theta x)\qquad (\theta^{-1}>a) \tag{4.8}$$

Then

$$\sigma_1(q,p)=\frac{H(p)}{p\theta+q}\qquad \left(H(p)=\int_0^{\infty}h(t)\,e^{-pt}dt\right)$$

Carrying out the integrations (4.6) and (4.7) by means of the residue theorem and inverting the result with respect to p, we obtain

$$\begin{aligned}
R(t)&=\left[C_5-\frac{\gamma}{b\theta}\right]a\int_0^t h(\tau)\,d\tau \qquad &&\left(C_5=\frac{\gamma k_0}{b\theta K(-b\theta)}\right)\\
M(t)&=\left[\left(\frac{\gamma}{b\theta}\right)^2-D_5\right]a^2\int_0^t (t-\tau)\,h(\tau)\,d\tau \qquad &&\left(D_5=\left(\frac{\gamma}{b\theta}\right)^2\frac{k_0-k_1 b\theta}{K(-b\theta)}\right)
\end{aligned} \tag{4.9}$$

5. We return to the problem of strip motion formulated in Section 3. The solution obtained was correct up to times for which the trace of the incident wave front had not reached the right edge of the strip. By the use of formulas (4.9), one may obtain the solution correct for

$$0\leqslant t\leqslant 2l\,/\,a \tag{5.1}$$

To do this it is necessary to take into account the effect of the interaction of the incident wave with the right edge of the strip. This may be done by adding to the motion produced by the reflection of a plane wave from the semi-infinite strip, the motion excited by the application to the boundary $x > 2l$ of the negative of the stresses which occur in the first motion.

The fronts of the incident, reflected waves and the fronts of the longitudinal waves produced on the edge of the strip are shown in Fig.7; the transverse wave fronts are not indicated in the figure.

If the time interval is restricted to (5.1), then it is sufficient to consider

$$\sigma_{yy}(x,t)=h(x,t)=-\frac{G(b\theta)}{\sqrt{1-a^2\theta^2}}\,a\rho v_0\,(t-\theta x-2l\theta) \tag{5.2}$$

This follows from (1.21) if one takes into account the fact that the trace of the incident front meets the left edge at $t = 0$.

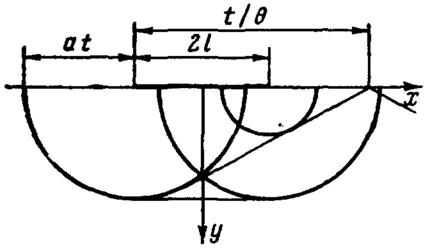

Fig. 7.

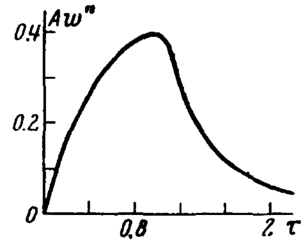

Fig. 8.

In accordance with (4.9), it is necessary to add to the expressions for the resultant (3.4) and the moment (3.5) the following

$$\Delta R(t) = -\frac{G(b\theta)}{\sqrt{1-a^2\theta^2}} C_5 (\lambda + 2\mu) v_0 (t - 2l\theta)$$
$$\Delta M(t) = \frac{G(b\theta)}{\sqrt{1-a^2\theta^2}} \left[C_5 a^2 \rho l v_0 (t - 2l\theta) + D_5 a^3 \rho \int_0^{t-2l\theta} v_0(\tau)\, d\tau \right] \tag{5.3}$$

These additions also take into consideration the fact that stresses of the type in the first term of (1.9) act only under the strip.

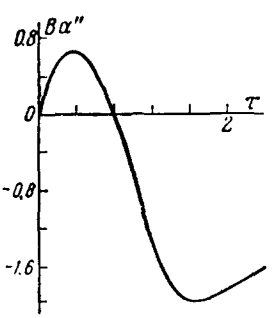

Fig. 9.

The computation (5.3) leads to the fact that on the right sides of (3.6) and (3.7) new terms appear which, for $t > 2l\theta$, are different from zero.

When $v_0(t)$ is chosen in the form of (3.10), these terms will be of the form

$$\frac{p_0}{a\rho}\left(\frac{a}{l}\right)^2 C_5 M (t - 2l\theta), \qquad -\frac{p_0}{a\rho}\left(\frac{a}{l}\right)^3 \frac{1}{l} 3 D_5 M$$

For $2l\theta < t < 2l/a$, this implies that, from the accelerations of translational and rotational motion computed by formulas (3.11) and (3.12), it is necessary to subtract the expressions which result from the same formulas when C_4 and D_4 are replaced by C_5 and D_5 and the argument t is changed to $t - 2l\theta$.

The graph of the translational motion acceleration shows that if $a\theta < 1/2$, then its maximum value is attained at the instant of appearance of the trace of the incident wave at the right edge of the strip and is of magnitude $w''(t)$ from (3.12), where $\tau = 2\, a\theta$.

In Figs. 8 and 9 are shown the dependence of w'' and α'' on $\tau = at/l$, computed under the same assumptions as in Figs. 5 and 6. The meaning of the notations A and B is also indicated above.

Below we give the dependence of C_5, D_5 on $a\theta$ for $\gamma = 0.6$.

TABLE 3.

$a\theta = 0.1$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$C_5 = 9.802$	4.800	3.130	2.310	1.813	1.486	1.251	1.079	0.9467
$D_5 = 114.0$	28.58	12.71	7.159	4.588	3.193	2.350	1.808	1.434

6. In studying the motion of the strip for $t < 2l/a$, we in fact used the solutions for the semi-infinite stamp and the reflection of a plane wave from a semi-infinite strip lying on an elastic half-space. Both of these led to self-modelling problems, which also explains the simplicity of the results. The construction of a solution valid over a large interval of time requires the solution of the problem of the interaction of the wave formed, say, on the left end of the strip with the free boundary found behind the right end of the strip.

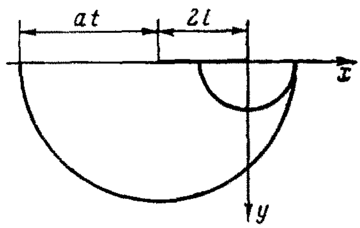

Fig. 10.

We locate the origin of coordinates on the left end of the strip, with the x-axis directed toward the right and the y-axis directed downward into the medium.

Let a wave be formed on the left end of the strip, producing stresses under the strip whose transform is of the form

$$\sigma_{yy}^{\circ}(x, p) = \int_{v_n} Q_n(M_n, p) \exp\left\{-\frac{spx}{b}\right\} dv_n \tag{6.1}$$

Here v_n is a certain volume in the n-dimensional space with coordinates $s_1, s_2, \ldots, s_n$, $Q_n(M_n, p)$ is a function of a point in the space and of the parameter p.

We shall assume that the resultant and moment of these stresses is known.

At the time $t = 2l/a$, the wave reaches the right end, upon which a new wave is formed.

For simplicity, only the fronts of the longitudinal waves are shown in Fig. 10.

The motion which is excited in the medium may be represented as the sum of a motion in the "incident" wave and one in the "reflected" wave, which should be such that its stresses, for $x \geqslant 2l$, are equal and of opposite sign to those of (6.1). Thereby, it will turn out that for $x \geqslant 2l$ the boundary is free of stress. In order to construct such a wave, it is necessary to use the results of Section 4.

We displace the origin of coordinates to the point $x = 2l$. Taking into account the above, it is necessary, for $x > 0$, to apply (in the new coordinates) a stress $h(x, t)$ such that

$$h^\circ(x, p) = \int_0^\infty e^{-pt} h(x, t)\, dt = -\int_{v_n} Q_n(M_n, p) e^{-u} dv_n \qquad (u = (2l + x) s_n p / b) \quad (6.2)$$

In accordance with (4.4)

$$\sigma_1(q, p) = -\frac{b}{p} \int_{v_n} Q_n(M_n, p) \exp\left\{-\frac{2s_n pl}{b}\right\} \frac{dv_n}{s_n + s} \qquad (6.3)$$

Carrying out the integrations determining $\sigma_{yy}^\circ(x, p)$ in accordance with (4.3) and (4.5), we obtain

$$\sigma_{yy}^\circ(x, p) = \int_\gamma^\infty ds \int_{v_n} Q_{n+1}(M_{n+1}, p)\, e^{spx/b}\, dv_n \qquad (6.4)$$

Here

$$Q_{n+1} = -Q_n \frac{sN(s)}{(s + s_n) K(-s_n)} \exp\left\{-\frac{2s_n pl}{b}\right\} \qquad (6.5)$$

The formulas (6.1) and (6.4) have the identical structure, and hence (6.4) and (6.5) are recurrence formulas for the formation of the transforms of the stresses in waves reflected from the ends of the strip

With the aid of (4.6) and (6.3), we obtain for the inverse of the resultant of the stresses (6.4)

$$R_{n+1}^\circ(p) = k_0 \frac{b}{p} \int_{v_n} Q_n(M_n, p) \exp\left\{-\frac{2s_n pl}{b}\right\} [1 - K^{-1}(-s_n)] \frac{dv_n}{s_n}$$

In order to compute the addition to the resultant $\Delta_n R^\circ(p)$ which is introduced by the reflection from the edge of the strip, one cannot restrict oneself to the expression introduced above, since a part of the resultant of the stresses (6.1) equilibrates (6.2). Taking this into account, we obtain

$$\Delta_n R^\circ(p) = -k_0 \frac{b}{p} \int\limits_{v_n} \frac{Q_n(M_n, p)}{s_n K(-s_n)} \exp\left\{-\frac{2s_n pl}{b}\right\} dv_n \tag{6.6}$$

Repeating similar arguments, we find from (4.7)

$$\Delta_n M^\circ(p) = -\left(\frac{b}{p}\right)^2 \int\limits_{v_n} \left(\frac{k_0}{s_n^2} - \frac{k_1}{s_n}\right) \frac{Q_n(M_n, p)}{K(-s_n)} \exp\left\{-\frac{2s_n pl}{b}\right\} dv_n \tag{6.7}$$

We describe now what will occur upon successive "reflections from the boundaries" of the wave whose stresses have the transform given by (1.9).

In this case v_1 is the interval $\gamma \leqslant s_1 \leqslant \infty$

$$Q_1 = \frac{a\rho}{k_0} pF(p) N(s_1)$$

Then from (6.6) it follows

$$\Delta_1 R^\circ(p) = -\gamma a^2 \rho F(p) \int\limits_\gamma^\infty P_1(s_1) \exp\left\{-\frac{2s_1 pl}{b}\right\} ds_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta_n R^\circ(p) = (-1)^n \gamma a^2 \rho F(p) \int\limits_{v_n} \Pi_n(M_n) E_n(M_n, p) \frac{dv_n}{s_1 s_n}$$

where the volume v_n is such that $s_k \geqslant \gamma$, $k = 1, 2, \ldots, n$

$$\Pi_n(M_n) = \prod_{k=1}^{k=n} P_k(s_k), \qquad E_n(M_n, p) = \exp\left\{-\frac{2pl}{b}\sum_{k=1}^{n} s_k\right\}$$
$$P_k(s_k) = \frac{s_k}{s_k + s_{k-1}} \frac{N(s_k)}{K(-s_k)}, \qquad P_1(s_1) = \frac{N(s_1)}{K(-s_1)} \tag{6.8}$$

Carrying out the inversion with respect to p, we obtain

$$\Delta_1 R(t) = -a^2 \rho \gamma \int\limits_\gamma^{bt/2l} P_1(s_1) f\left(t - \frac{2s_1 l}{b}\right) \frac{ds_1}{s_1}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots \tag{6.9}$$
$$\Delta R_n(t) = (-1)^n a^2 \rho \gamma \int\limits_{v_n} \Pi_n(M_n) f\left(t - \frac{2l}{b}\sum_{k=1}^{n} s_k\right) \frac{dv_n}{s_1 s_n}$$

Here v_n is a volume in the n-dimensional space $s_1, \ldots, s_n$ such that

$$s_k \geqslant \gamma, \qquad \sum_{k=1}^{n} s_k \leqslant \frac{bt}{2l} \tag{6.10}$$

By a change of variables these expressions may be brought to the form

$$\Delta_n R\ (t) = (-1)^n\, 2a^3\rho l^{-1} \int_0^{T_n} f\ (\tau)\ S_n\ (t-\tau)\ d\tau \qquad \left(T_n = t - \frac{2ln}{a}\right) \tag{6.11}$$

Then in the case of translational motion of the strip, we have, with account of (1.11) and (1.12), the following for the resultant of the stresses acting on the strip

$$R\ (t) = -2a\rho l f'\ (t) - C_1 a^2 \rho f\ (t) + 4a^3 \rho l^{-1} \sum_{n=1}^{m} (-1)^n \int_0^{T_n} f\ (\tau)\ S_n\ (t-\tau)\ d\tau \tag{6.12}$$
$$(m = E\ (at/2l))$$

It is not difficult to discern that the structure of (6.12) is such that, after substitution of $R(t)$ into (2.1) and into the first equation (3.3) for the determination of $v_1(t)$ and $w(t)$, ordinary differential equations with constant coefficients will be obtained at each step. Hence the problem that has been posed may be solved by quadratures.

We consider now the reflection from the boundaries of a wave the transform of whose stresses is given by the second term of (1.17)

$$Q_1 = \frac{a^2\rho}{k_0^2} A\ (p) \left(k_1 - \frac{k_0}{s_1}\right) N\ (s_1) \tag{6.13}$$

and v_1 is the semi-infinite interval $\gamma \leqslant s_1$.

Then from (6.5), (6.6), (6.7) and (6.13) it follows that

$$\Delta_n R\ (t) = (-1)^n \gamma a^2 \rho \int_{v_n} \left(\frac{k_1}{k_0} - \frac{1}{s_1}\right) \Pi_n\ (M_n)\, \alpha_1 \left(t - \frac{2l}{b} \sum_{k=1}^{n} s_k\right) \frac{dv_n}{s_1 s_n} \tag{6.14}$$

$$\Delta_n M\ (t) = (-1)^n \gamma^2 a^3 \rho \int_{v_n} \left(\frac{k_1}{k_0} - \frac{1}{s_1}\right) \left(\frac{k_1}{k_0} - \frac{1}{s_n}\right) \Pi_n\ (M_n)\, \alpha_2 \left(t - \frac{2l}{b} \sum_{k=1}^{n} s_k\right) \frac{dv_n}{s_1 s_n}$$

The integration is carried out over the volume (6.10). In addition, we have introduced the notation

$$\alpha_1\,(\xi) = \int_0^{\xi} \alpha\ (t)\ dt, \qquad \alpha_2\ (\xi) = \int_0^{\xi} (\xi - t)\ \alpha\ (t)\ dt$$

To obtain an expression for the increase of the moments relative to the middle of the strip for a rotation through the angle $\alpha(t)$, it is necessary to proceed in the same way as in Section 3 for the derivation of formula (3.6). In this, one must use formulas (6.14) and (6.9). In order to generalize equation (3.7), it is necessary to compute the increase of the resultant and moment of the stresses (1.21).

Then, after some transformations originating from (6.11), at each step one obtains ordinary differential equations of fourth order for $\alpha(t)$ as well as for $w(t)$. From these $\alpha(t)$ may be found at an arbitrary instant of time.

7. As is seen from (1.3), (1.10), (6.8) and (6.9), the functions in (6.12) are extremely complicated. Hence it is desirable to determine, even if approximately, a representation of the role of the supplementary terms in (6.12). To do this, we approximate $\varphi(\xi)$ from (1.3) by the function $\psi(\xi)$

$$\psi(\xi) = B(\gamma)\frac{\sqrt{(1-\xi)(\xi-\gamma)}}{\xi-\xi_0(\gamma)}$$

The constants $B(\gamma)$ and $\xi_0(\gamma)$ are to be chosen so that $\psi = \varphi$ at $\xi = 1/\sqrt{2}$, where $\varphi(\xi)$ has a maximum equal to $\pi/2$. As a result, we obtain a function which coincides with $\varphi(\xi)$ at three points and, moreover, is such that its derivatives at these points also coincide. For $0.4 < \gamma < 0.65$ the accuracy of this approximation is good.

Now, $g(s)$, $K(s)$, $N(s)$ and $P(s)$ may be expressed in terms of elementary functions. For $\gamma = 0.6$, $\Delta_1 R(t)$ and $\Delta_2 R(t)$ were computed for the case $f(t) = f_0$ const. For $2l/a < t < 8l/a$ it turned out that $\Delta_1 R(t)$ does not exceed $4 \times 10^{-3}(\lambda + 2\mu)f_0$, and $\Delta_2 R(t)$ for $4l/a < t < 8l/a$ does not exceed $5 \times 10^{-4}(\lambda + 2\mu)f_0$. Comparing this with the fact that for $0 < t < 2l/a$

$$R(t) = -2a\rho l f_0 \delta(t) - 0.4343(\lambda + 2\mu) f_0$$

(for large periods it is necessary to add $\Delta_1 R(t)$, $\Delta_2 R(t)$, etc. to this expression), one may conclude that the additions introduced into the expressions for the resultant in the stamp problem when reflections from the ends are taken into account, will not be large when $f(t)$ changes slowly for $t > l/a$.

This gives some basis for asserting that the solution of the strip motion problem and the reductions in Sections 2, 3 and 5 are certain approximation that is suitable over a large interval of time, as has been indicated.

The author would like to thank N.V. Zvolinskii for his great interest in this work.

BIBLIOGRAPHY

1. Flitman, L.M., Dinamicheskaia zadacha o shtampe na uprugoi poluploskosti (Dynamic problem of the stamp on an elastic half-plane). *PMM*, Vol. 23, No. 4, 1959.

2. Galin, L.A., Udar po tverdomu telu nakhrodiashchemusia na poverkhnosti szhimaemoi zhidkosti (Impact on a rigid body located on the surface of a compressible fluid). *PMM* Vol. 11, No. 5, 1947.

3. Afanas'ev, E.F., Difraktsiia nestatsionarnoi volny davleniia na podvizhnoi plastine (Defraction of a nonstationary pressure wave on a moving plate). *PMM*, Vol. 26, No. 1, 1962.

4. Flitman, L.M., Ob odnoe smeshannoi kraevoi zadache dlia volnovogo uravnenia (On a mixed boundary value problem for the wave equation). *PMM*, Vol. 22, No. 6, 1958.

5. Titchmarsh, E., *Introduction to Fourier Integrals.* (Russian Translation). Gostekhizdat, 1948.

6. Fock. V.A., O nekotorykh integral'nykh uravneniiakh matematicheskoi fiziki (On certain integral equations of mathematical physics). *Matem. sb.*, Vol. 14(56), No. 1-2, 1943.

Translated by E.E.Z.